

The Threshold for Fault-Tolerant Quantum Computation

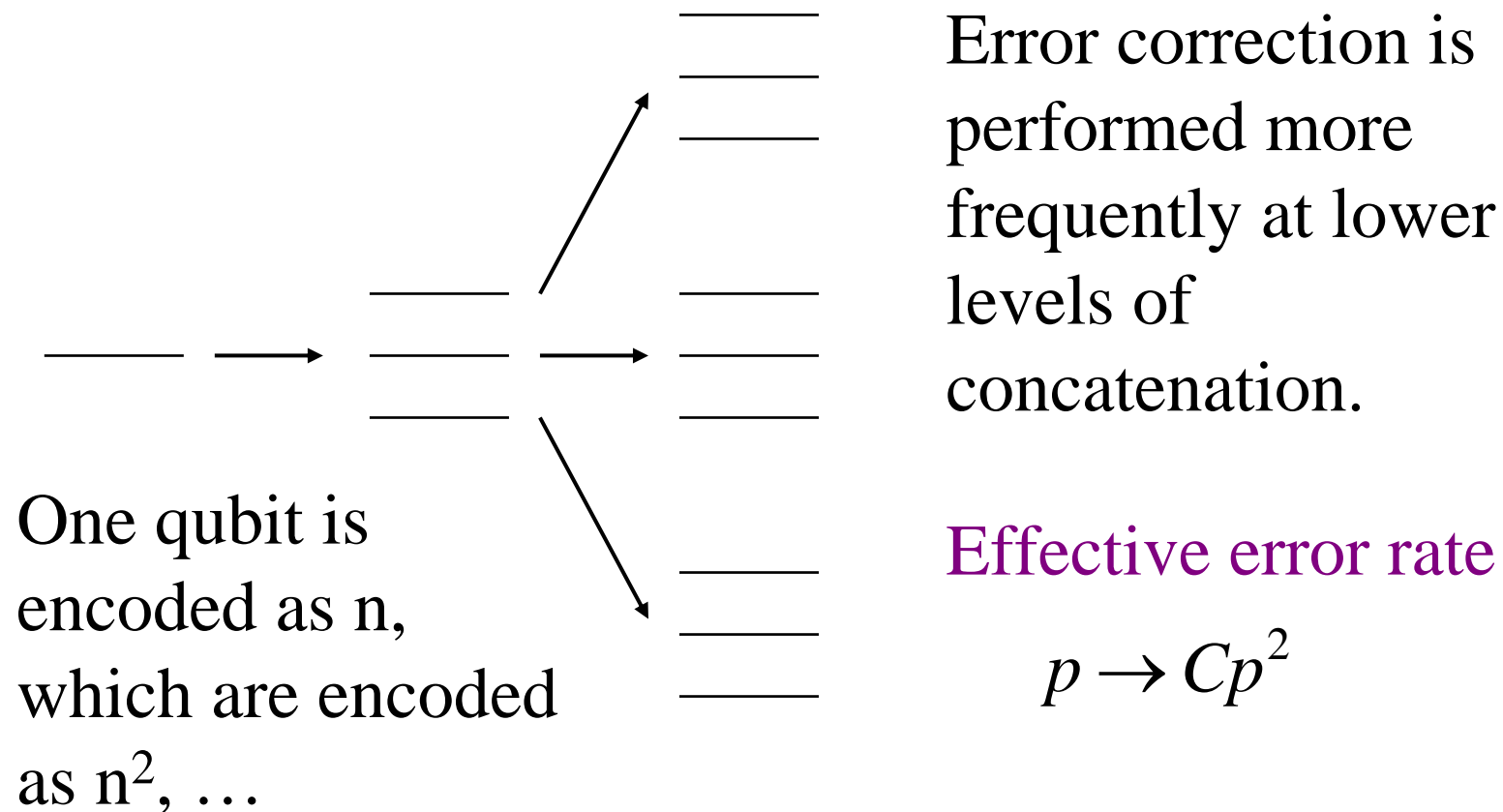
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Basics of Fault-Tolerance

- The purpose of fault-tolerance is to enable reliable quantum computations when the computer's basic components are unreliable.
- To achieve this, the qubits in the computer are encoded in blocks of a quantum error-correcting code, which allows us to correct the state even when some qubits are wrong.
- A fault-tolerant protocol prevents catastrophic error propagation by ensuring that a single faulty gate or time step produces only a single error in each block of the quantum error-correcting code.

Concatenated Codes

Threshold for fault-tolerance proven using concatenated error-correcting codes.



Threshold for Fault-Tolerance

Theorem: There exists a threshold p_t such that, if the error rate per gate and time step is $p < p_t$, arbitrarily long quantum computations are possible.

Proof sketch: Each level of concatenation changes the effective error rate $p \rightarrow p_t (p/p_t)^2$. The effective error rate p_k after k levels of concatenation is then

$$p_k < p_t (p/p_t)^{2^k}$$

and for a computation of length T , we need only $\log(\log T)$ levels of concatenation, requiring $\text{polylog}(T)$ extra qubits, for sufficient accuracy.

Determining the Threshold Value

There are three basic methodologies used to determine the value of the threshold:

- **Numerical simulation:** Randomly choose errors on a computer, see how often they cause a problem. Tends to give high threshold value, but maybe this is an overestimate; only applies to simple error models.
- **Rigorous proof:** Prove a certain circuit is fault-tolerant for some error rate. Gives the lowest threshold value, but everything is included (up to proof's assumptions).
- **Analytic estimate:** Guess certain effects are negligible and calculate the threshold based on that. Gives intermediate threshold values.

History of the Threshold

Golden

Age (1996)

Shor (1996) - FT protocols

Aharonov, Ben-Or
(1996) - threshold
proof

Knill, Laflamme (1996)
storage threshold

Kitaev (1996-...)
topological FT,
threshold

Zalka (1996)
simulation

K, L, Zurek (1996)
threshold

Dark
Ages

Other
simulations

G, Preskill
higher value

Dennis et al. (2001)
topological threshold

Aliferis, G, Preskill
(2005) - simple proof

Knill (2004),
Reichardt (2004)
very high threshold

Reichardt (2005)
d=3 proof

Renaissance (2004-)

Local gates, specific systems, ...

Requirements for Fault-Tolerance

1. Low gate error rates.
2. Ability to perform operations in parallel.
3. A way of remaining in, or returning to, the computational Hilbert space.
4. A source of fresh initialized qubits during the computation.
5. Benign error scaling: error rates that do not increase as the computer gets larger, and no large-scale correlated errors.

Additional Desiderata

1. Ability to perform gates between distant qubits.
2. Fast and reliable measurement and classical computation.
3. Little or no error correlation (unless the registers are linked by a gate).
4. Very low error rates.
5. High parallelism.
6. An ample supply of extra qubits.
7. Even lower error rates.

Threshold Values

Computed threshold value depends on **error-correcting code, fault-tolerant circuitry, analysis technique**. Assume for now that all additional desiderata are satisfied.

- Concatenated 7-qubit code, standard circuitry:
 - Threshold $\sim 10^{-3}$ (various simulations)
 - Threshold $\sim 3 \times 10^{-5}$ (proof: Aliferis, Gottesman, Preskill, quant-ph/0504218; also Reichardt, quant-ph/0509203)
- Best known code: 25-qubit Bacon-Shor code
 - Threshold $\sim 2 \times 10^{-4}$ (proof, Aliferis, Cross, quant-ph/0610063)

Ancilla Factories

Best methods trade extra ancilla qubits for error rate:

Ancilla factories create complex ancilla states to substitute for most gates on the data. Errors on ancillas are less serious, since bad ancillas can be discarded safely.

Extreme case: Create all states using error-detecting codes, ensuring a low basic error rate but very high overheads (e.g. 10^6 or more physical qubits per logical qubit) -- Knill, quant-ph/0404104, Reichardt, quant-ph/0406025.

- Simulations: threshold $\sim 1\%$ or higher.
- Provable threshold $\sim 10^{-3}$? (forthcoming)

Local Gates

Proof that threshold still exists with local gates: Gottesman, quant-ph/9903099; Aharonov, Ben-Or, quant-ph/9906129.

We are starting to understand the value of the threshold in this case:

- With concatenation, in 2D, proven threshold of $\sim 10^{-5}$ (Svore, Terhal, DiVincenzo, quant-ph/0604090)
- Almost 2D, w/ topological codes & cluster states, simulated threshold of $\sim 6 \times 10^{-3}$ (Raussendorf, Harrington, quant-ph/0610082)
- Almost 1D: simulation gives threshold of $\sim 10^{-6}$ (Szkopek et al., quant-ph/0411111)

Non-Markovian Errors

What happens when the environment has a memory?

- Questioning fault-tolerance for non-Markovian environments: Alicki, Horodecki³ (quant-ph/0105115), Alicki, Lidar, Zanardi (quant-ph/0506201)
- Proof of fault-tolerant threshold with single-qubit errors and separate environments for separate qubits: Terhal, Burkhard (quant-ph/0402104)
- Proof of fault-tolerant threshold with shared environment: Aliferis, Gottesman, Preskill (quant-ph/0504218)
- With 2-qubit errors: Aharonov, Kitaev, Preskill (quant-ph/0510231)
- Unbounded Hamiltonians (spin boson model)? See Terhal, Burkhard and Klesse, Frank (quant-ph/0505153)

The Future of Fault-Tolerance

Industrial Age

Experimental FT

Ancilla
factories

Understanding
of resource
tradeoffs

Efficient fault-
tolerance

Large quantum
computers!

Quantum
Information Age

